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BUCKLING AND POSTBUCKLING BEHAVIOR OF STEEP COMPRESSIBLE ARCHES

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Abstract—The bifurcation buckling and postbuckling behavior of steep, compressible, circular arches is examined. The arches are loaded with a uniform constant directional pressure and may be either pinned or clamped. The development is based on Koiter's theory. Two different arch theories are used so as to facilitate a study of bending in the prebuckling state. It is shown that clamped arches are always unstable after bifurcation, while pinned arches exhibit a transition from unstable to stable behavior as a semi-circular arch is approached. The results are also compared to results obtained using shallow arch theory and the comparison is reasonably good for moderately steep arches. The effect of middle surface extensibility (compressibility) and of the prebuckling bending is virtually undetectable.

1. INTRODUCTION

THIS paper treats the bifurcation buckling and postbuckling behavior of steep, compressible, circular arches under constant directional ("dead") pressure. A linearized treatment of the buckling problem for inextensible arches has been given by Chwalla and Kollbrunner [1]. The extensional buckling problem has been treated by Kämmel [2] and Dym [3], while the relationship between ring buckling and that of semi-circular arches has been discussed by Singer and Babcock [4].

Implicit closed form (in terms of elliptical integrals) solutions for the postbuckling behavior of inextensional circular rings have been given by Lévy [5] and Carrier [6]. Schmidt and Da Deppo [7] have given a rather complete set of arch equations, while presenting some results for eccentrically loaded inextensional arches. Huddleston [8, 9] has developed a numerical algorithm for a complete set of curved-beam equations and has investigated the behavior of a number of arches of varying geometry under centrally applied concentrated loads.

The present work presents analyses of the effect of prebuckling bending and of compressibility on the buckling and postbuckling behavior of steep arches. The analyses are based on the theory advanced by Koiter [10], although the formalism developed by Budiansky and Hutchinson [11–13] is used here.

Two different kinematic descriptions are used, the difference residing in the expression for the curvature change. While the difference is small, it allows the inclusion of bending during the prebuckling state (in one case), so that the significance of such bending can be assessed.

Both clamped and pinned arches are considered. In addition, results for complete rings can be deduced from the present results by considering semi-circular arches. One interesting phenomenon that then arises is the transition from the stable postbuckling behavior of rings to the instability of steeper arches.

The critical pressures and postbuckling coefficients are also compared to results obtained using the shallow arch kinematic relations. Good agreement is demonstrated for moderately steep arches.

The basic kinematic relations, differential equations and postbuckling relations are developed and displayed in the text. Some of the details of the buckling eigenvalue problems and of the various kinematic solutions are given in the Appendices.

2. BASIC EQUATIONS

Two sets of kinematic relations will be used to describe the arch behavior. The first is a subset of the nonlinear shell equations of Sanders [14] and is referred to as the set of full nonlinear ring equations (FNR). The second set represents a modification in the curvature change to a fairly common form, and is referred to as the set of modified curvature ring equations (MCR). The constitutive relations are common to both sets of ring equations and are

$$N = \frac{1}{H}(e+\tfrac{1}{2}\chi^2) \tag{1a}$$

$$M = K \tag{1b}$$

where N and M are dimensionless stress and moment resultants, e and χ are the linear in-plane strain and the rotation and K is the curvature change. When expressed in terms of the tangential displacement v and the radial displacement w these kinematic quantities take the forms

$$\text{FNR}:\ e = v'-w, \qquad \chi = w'+v, \qquad K = -\chi' \tag{2a}$$

$$\text{MCR}:\ e = v'-w, \qquad \chi = w'+v, \qquad K = -(w''+w). \tag{2b}$$

The primes denote differentiation with respect to the arch coordinate ϕ, which along with the displacements has been nondimensionalized with respect to the arch radius R, while the stress and moment resultants have been rendered dimensionless by dividing by (EI/R^2) and (EI/R), respectively. This results in a thickness ratio (compressibility parameter) being introduced in equation (1a), i.e.

$$H = \frac{1}{12}\left(\frac{h}{R}\right)^2. \tag{3}$$

Further, a dimensionless (uniform) applied pressure p will be defined as

$$p = \frac{qR^3}{EI} \tag{4}$$

where q is the load (uniform here)/unit length of arch.

A variational (virtual work) statement of the problem can be given as

$$\int_{-\alpha}^{\alpha} [N\delta e+N\chi\delta\chi+M\delta K-p\delta w]\,\mathrm{d}\phi = 0. \tag{5}$$

Equation (5) is the starting point for the Budiansky–Hutchinson formalism.

The exact nonlinear differential equations that are the Euler–Lagrange equations of the variational statement (5) are:

$$\text{FNR}: \quad \begin{aligned} &N' - N\chi - M' = 0 \\ &-N - (N\chi)' - M'' = p \end{aligned} \tag{6a}$$

and

$$\text{MCR}: \quad \begin{aligned} &N' - N\chi = 0 \\ &-N - (N\chi)' - M'' - M = p. \end{aligned} \tag{6b}$$

For clamped and pinned arches the corresponding boundary conditions are, respectively,

$$v = w = w' = 0 \quad \text{at } \phi = \pm\alpha \tag{7a}$$

and

$$v = w = 0 \quad \text{and} \quad M = K = 0 \quad \text{at } \phi = \pm\alpha. \tag{7b}$$

3. BUCKLING AND POSTBUCKLING

In this section the initial postbuckling "load–deflection" relation of the arch will be given, following the asymptotic analysis outlined in the very lucid exposition of Budiansky [13]. Only asymmetric buckling of "steeper" arches is considered here since for very shallow arches symmetric snap-through buckling can take place at lower pressures. However, this snap type of buckling cannot be analyzed with the Koiter theory.

The following expansions are introduced:

$$\begin{aligned} v &= pv_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \\ w &= pw_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \\ e &= pe_0 + \varepsilon e_1 + \varepsilon^2 e_2 + \ldots \\ \chi &= p\chi_0 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \ldots \\ K &= pK_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \ldots \\ N &= -N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots \\ M &= M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \ldots \end{aligned} \tag{8}$$

Here ε is a small parameter such that when $\varepsilon \to 0$, then $p \to p_{cr}$, the bifurcation buckling pressure. The quantity χ_0 shall be henceforth taken as zero, with $K_0 \neq 0$, so that a linear prebuckling state that includes bending is assumed. The effect of bending in the prebuckling state can thus be assessed. A uniform membrane state can be obtained, prior to buckling, by requiring $K_0 = 0$, as must be done for the FNR equations.

If the expansions (8) are introduced into the nonlinear differential equations (6), the equations governing the prebuckling, buckling and postbuckling deformation are easily sorted out. These equations are:

Prebuckling†

$$\text{FNR}: \; N_0' = (p)' = 0. \tag{9a}$$

$$\text{MCR}: \; N_0' \equiv (p\psi)' = 0, \qquad M_0'' + M_0 - N_0 = -p \tag{9b}$$

Buckling

$$\text{FNR}: \; N_1' + p_{cr}\chi_1 - M_1' = 0, \qquad M_1'' + N_1 - p_{cr}\chi_1' = 0. \tag{10a}$$

$$\text{MCR}: \; N_1' + \psi p_{cr}\chi_1 = 0, \qquad M_1'' + M_1 + N_1 - \psi p_{cr}\chi_1' = 0 \tag{10b}$$

Postbuckling

$$\text{FNR}: \; \begin{aligned} N_2' + p_{cr}\chi_2 - M_2' &= N_1\chi_1 \\ M_2'' + N_2 - p_{cr}\chi_2' &= -(N_1\chi_1)'. \end{aligned} \tag{11a}$$

$$\text{MCR}: \; \begin{aligned} N_2' + p_{cr}\psi\chi_2 &= N_1\chi_1 \\ M_2'' + M_2 + N_2 - p_{cr}\psi\chi_2' &= -(N_1\chi_1)'. \end{aligned} \tag{11b}$$

It can be further shown (see Refs. [13, 17]) that the asymptotic expansions (8), when introduced into the variational statement (5), imply a load-deflection relationship of the form

$$\frac{p}{p_{cr}} = 1 + a\varepsilon + b\varepsilon^2 + \ldots \tag{12}$$

where

$$a = \frac{3}{2\psi p_{cr}} \frac{\int_{-\alpha}^{\alpha} N_1 \chi_1^2 \, \mathrm{d}\phi}{\int_{-\alpha}^{\alpha} \chi_1^2 \, \mathrm{d}\phi} = \frac{3}{2H\psi p_{cr}} \frac{\int_{-\alpha}^{\alpha} e_1 \chi_1^2 \, \mathrm{d}\phi}{\int_{-\alpha}^{\alpha} \chi_1^2 \, \mathrm{d}\phi} \tag{13a}$$

$$b = \frac{1}{\psi p_{cr}} \frac{\int_{-\alpha}^{\alpha} [2N_1\chi_1\chi_2 + N_2\chi_1^2] \, \mathrm{d}\phi}{\int_{-\alpha}^{\alpha} \chi_1^2 \, \mathrm{d}\phi} = \frac{1}{H\psi p_{cr}} \frac{\int_{-\alpha}^{\alpha} [2e_1\chi_1\chi_2 + (e_2 + \frac{1}{2}\chi_1^2)\chi_1^2] \, \mathrm{d}\phi}{\int_{-\alpha}^{\alpha} \chi_1^2 \, \mathrm{d}\phi} \tag{13b}$$

and where p_{cr} is defined either as the lowest eigenvalue of equations (10) or by the corresponding Rayleigh quotient:

$$\psi p_{cr} = \frac{\int_{-\alpha}^{\alpha} (N_1 e_1 + M_1 K_1) \, \mathrm{d}\phi}{\int_{-\alpha}^{\alpha} (\chi_1)^2 \, \mathrm{d}\phi}. \tag{14}$$

Recall that according to this theory, if $a \neq 0$, then the structure will be in an unstable state in the postbuckling range and it will also be imperfection sensitive. If $a = 0$, $b \neq 0$, then the postbuckling stability and the imperfection sensitivity vary as the arithmetic sign of b.

In the present result it appears at first glance that the existence of a stable or unstable equilibrium state in the postbuckling range depends on whether or not inextensibility is assumed. For from equation (13a), if $N_1(\phi)$ is non-zero, then the coefficient a might be. However, it seems rather clear that for asymmetric buckling, the membrane stress resultant due to buckling must itself be an odd function of ϕ and hence $a \equiv 0$. It is also arguable

† In equation (9b) a parameter ψ is introduced that reflects the presence of the linear bending approximation to the prebuckling state.

that since the pressure-loaded arch is a symmetrical structure which bifurcates asymmetrically, the coefficient a must be zero.

Details of the solutions of equations (9)–(11) and of evaluation of the coefficients (13) are given in the Appendices.

4. NUMERICAL RESULTS

The numerical results for the critical buckling pressures and for the postbuckling coefficient b are displayed in Tables 1–7. Details of the solution are given in the Appendices.

The critical pressures (Tables 1, 2) appear to be almost completely insensitive to compressibility (H) and to the bending (MCR) or membrane (FNR) nature of the prebuckling solution. In fact, the results are not significantly different from results reported for inextensional buckling based upon a membrane prebuckling state [1, 3]. Further, if the parameter H is set to zero, the eigenvalue problems (A.4, A.5, B.9, B.10) reduce to the transcendental equations of Chwalla and Kollbrunner [1].

The magnitude of the bending in the prebuckling state can be further assessed by noting that the bending moment is proportional to $p(1-\psi)$.† Some typical values of the parameter ψ are displayed in Table 3. These results clearly indicate the smallness of the bending resultant. After buckling, the only effect of ψ is through its (negligible) effect on p_{cr}.

TABLE 1. CRITICAL BUCKLING PRESSURES FOR PINNED ARCHES

	p_{cr} FNR		MCR	
α (deg.)	$h/R = 1/10$	$h/R = 1/1000$	$h/R = 1/10$	$h/R = 1/1000$
90	3·26923	3·27124	3·27053	3·27125
80	4·51287	4·51449	4·51359	4·51449
70	6·21586	6·21707	6·21752	6·21707
60	8·72629	8·72712	8·73700	8·72712
50	12·78071	12·78124	12·83972	12·78125
40	20·14096	20·14128	20·46900	20·14131

TABLE 2. CRITICAL BUCKLING PRESSURES FOR CLAMPED ARCHES

	p_{cr} FNR		MCR	
α (deg.)	$h/R = 1/10$	$h/R = 1/1000$	$h/R = 1/10$	$h/R = 1/1000$
90	9.00000	9·00000	9·03207	9·00000
80	11·33095	11·33135	11·40456	11·33136
70	14·61636	14·61805	14·79662	14·61806
60	19·58254	19·58671	20·07098	19·58676
50	27·73864	27·74726	29·27890	27·74742
40	42·68493	42·70190	48·80510	42·70251

† See equation (B.2).

TABLE 3. BENDING PARAMETER FOR PREBUCKLING SOLUTION (MCR): $M_0 \sim p(1-\psi)$

α (deg.)	Pinned $h/R = 1/10$	Pinned $h/R = 1/1000$	Clamped $h/R = 1/10$	Clamped $h/R = 1/1000$
	ψ			
90	1·000000	1·000000	0·995620	1·000000
80	0·999927	1·000000	0·992719	0·999999
70	0·999562	1·000000	0·986995	0·999999
60	0·998397	1·000000	0·974852	0·999997
50	0·994872	0·999999	0·946604	0·999994
40	0·983332	0·999998	0·873872	0·999986

Typical results for the postbuckling coefficient b are shown in Tables 4 and 5. Recall that a positive coefficient denotes stability and a negative coefficient instability in the early postbuckling behavior. It is immediately clear that the coefficients are relatively insensitive to the effect of compressibility.† It may also be noted that for the clamped arches (Table 5) the values of the postbuckling coefficients are very close, when both arch theories are compared.

Perhaps the most interesting results are the values of the postbuckling coefficients for the pinned arches (Table 4). There is clearly a transition, as the arch becomes progressively steeper, from unstable to stable postbuckling behavior. This represents a transition from

TABLE 4. POSTBUCKLING COEFFICIENTS FOR PINNED ARCHES

α (deg.)	FNR $h/R = 1/10$	FNR $h/R = 1/1000$	MCR $h/R = 1/10$	MCR $h/R = 1/1000$
	b			
90	1·019302	1·011415	0·211386	0·211430
80	0·386367	0·381016	−0·232681	−0·229784
70	−0·484773	−0·484626	−1·227794	−1·212972
60	−2·210659	−2·184584	−3·414991	−3·350614
50	−6·556133	−6·370323	−8·719322	−8·407426
40	−20·298221	−18·804925	−24·564979	−22·521435

TABLE 5. POSTBUCKLING COEFFICIENTS FOR CLAMPED ARCHES

α (deg.)	FNR $h/R = 1/10$	FNR $h/R = 1/1000$	MCR $h/R = 1/10$	MCR $h/R = 1/1000$
	b			
90	−0·575536	−0·571700	−0·451288	−0·446700
80	−1·204257	−1·190227	−1·230642	−1·211721
70	−2·446248	−2·396215	−2·723509	−2·657893
60	−5·194701	−4·993075	−5·878286	−5·629916
50	−12·307776	−11·294352	−13·719296	−12·546408
40	−36·999547	−29·413727	−40·114950	−31·784643

† It is also seen that the coefficients are all of order unity. This is due to normalizing the maximum physical deflection to equal the radius. If the normalization were with respect to the thickness, each coefficient would be multiplied by $(h/R)^2$. See also Koiter's elastica analysis (Ref. [10], Section 613).

steep arch unstable bifurcation (see, for example, Schreyer and Masur [15]) to the stable postbuckling behavior of complete rings [5, 6]. That is, the behavior of a ring may be analyzed by considering a pinned semi-circular ($\alpha = \pi/2$) arch [4]. It is also of interest to note that for the pinned arches, the different arch theories produced quantitatively different results (although qualitatively the results are the same). This can be seen from the difference in the values of the coefficients, and also by noting a difference in the transition angle. The transition from unstable to stable behavior occurs at $\alpha \simeq 75°$ for the FNR solution, and at $\alpha \simeq 84°$ for the MCR solution.

The question of why these two arch theories produce very good agreement for the clamped arches and show discrepancies for the pinned arches may be answered by examining the boundary conditions. First, by inspection of equations (A.6), (A.7) and (B.11), (B.12), a considerable degree of similarity is observed in the displacement solutions to the two problems, and, in fact, the numerical values of the particular solution constants for the second order solution (C_i, C_{i+4}) are in very good agreement. The D_j are determined by satisfaction of the boundary conditions, whose formulation for the clamped case is identical [equations (7a)].

For the pinned case, the stress boundary condition takes the forms

$$\begin{aligned} \text{FNR: } & w_i'' + v_i' = 0 \\ \text{MCR: } & w_i'' + w_i' = 0 \end{aligned} \tag{15}$$

for the ith order solution. The definition of the linear in-plane strain e_i [equation (2a)] may be used to modify, for example, the first of equations (15) to yield:

$$\text{FNR: } w_i'' + w_i + e_i = 0. \tag{16}$$

Some relevant orders of magnitude—discernible from detailed examination of the displacement coefficients—of the e_i are given in the Appendices. Since the displacements are of order unity, and since the buckling in-plane strain e_1 is of order H, it is seen that the buckling stress boundary condition is virtually the same for both theories. For the second order problem, however, e_2 is of order unity, and thus there is a significant difference in the boundary conditions, therefore, in the constants D_i, and so in the postbuckling coefficients.

Tables 6 and 7 illustrate a comparison of some of the present results (MCR, $h/R = 1/1000$) with results obtained in a parallel investigation using shallow arch theory. In Ref. [16] it is shown that the critical pressures are given by

$$\begin{aligned} \text{Pinned: } p_{cr} &= \left(\frac{\pi}{\alpha}\right)^2 \left(1 + \frac{5}{8\lambda^2}\right)^{-1} = \left(\frac{\pi}{\alpha}\right)^2 \left[1 - \frac{5}{8\lambda^2} + \left(\frac{5}{8\lambda^2}\right)^2 + \dots\right] \\ \text{Clamped: } p_{cr} &= \left(1{\cdot}43\frac{\pi}{\alpha}\right)^2 \left(1 + \frac{15}{4\lambda^2}\right)^{-1} = \left(1{\cdot}43\frac{\pi}{\alpha}\right)^2 \left[1 - \frac{15}{4\lambda^2} + \left(\frac{15}{4\lambda^2}\right)^2 + \dots\right] \end{aligned} \tag{17}$$

and the postbuckling coefficients are

$$\begin{aligned} \text{Pinned: } b &= \frac{3(R/h)^2}{[1 - \lambda^2/1{\cdot}90]} = -\frac{3(1{\cdot}90)}{\alpha^4}\left[1 + \frac{1{\cdot}90}{\lambda^2} + \left(\frac{1{\cdot}90}{\lambda^2}\right)^2 + \dots\right] \\ \text{Clamped: } b &= \frac{1{\cdot}61(R/h)^2}{[1 - \lambda^2/5{\cdot}02]} = -\frac{(1{\cdot}61)(5{\cdot}02)}{\alpha^4}\left[1 + \frac{5{\cdot}02}{\lambda^2} + \left(\frac{5{\cdot}02}{\lambda^2}\right)^2 + \dots\right] \end{aligned} \tag{18}$$

TABLE 6. COMPARISON OF CRITICAL LOADS FROM SHALLOW ARCH ANALYSIS

α (deg.)	p_{cr}			
	Pinned		Clamped	
	MCR	Shallow	MCR	Shallow
90	3·27	4·00	9·00	8·19
80	4·51	5·06	11·33	10·35
70	6·22	6·61	14·62	13·52
60	8·73	9·00	19·59	18·40
50	12·78	12·96	27·75	26·55
40	20·14	20·25	42·70	41·45

TABLE 7. COMPARISON OF POSTBUCKLING COEFFICIENTS FROM SHALLOW ARCH ANALYSIS

α (deg.)	b			
	Pinned		Clamped	
	MCR	Shallow	MCR	Shallow
90	+0·211	−0·933	−0·447	−1·33
80	−0·230	−1·50	−1·212	−2·14
70	−1·213	−2·54	−2·658	−3·72
60	−3·350	−4·71	−5·630	−6·71
50	−8·407	−9·80	−12·546	−13·92
40	−22·521	−24·0	−31·785	−34·2

where $\lambda = \alpha^2(R/h)$ is the arch rise parameter introduced by Schreyer and Masur [15]. The numerical values displayed in Tables 6, 7 represent the leading terms in the expansions (17) and (18). The agreement for both the critical pressures and the postbuckling coefficients is seen to be reasonably good over the middle range of steep arches, i.e $40° < \alpha < 60°$. Of course, as mentioned earlier, the present analyses do not deal with symmetric snap-through, and so the discussion of when an arch is shallow enough for snap-through to govern is not dealt with (see Refs. [16, 3]).

5. CONCLUSIONS

This paper has presented a study of the buckling and postbuckling behavior of steep, compressible, circular arches. It has been shown that the effect of bending in the pre-buckling state and of compressibility is minimal. The postbuckling behavior of clamped arches has been shown to be unstable for the complete range of arch vertex examples considered. The pinned arch results demonstrate a transition from instability to the stability characteristic of complete rings as the arch semi-vertex angle approaches ninety degrees. The results found were essentially the same for both arch theories used, with some slight quantitative differences in the postbuckling behavior of the pinned arches. Finally, good correlation with corresponding shallow arch results has been indicated for the expected range of agreement.

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APPENDIX A

The procedure for obtaining the solutions is straightforward:

1. Integration of equations (9) to obtain the prebuckling solution.
2. Integration of equations (10) to obtain the buckling solution.
3. Formulation of the eigenvalue problems for the pinned and clamped arch critical pressures.
4. Normalization of the buckling mode so that the maximum radial displacement (dimensionless) is unity.
5. Integration of equations (11) to obtain the second order solution.
6. Evaluation (analytical) of the integrals in equation (13).

A digital computer was used to solve the transcendental equation for the eigenvalues, to obtain the normalized buckling displacement coefficients, to solve the linear algebraic equations to satisfy the boundary conditions for the second order solution, and to obtain numerical results from the analytical integrations of the integrals in equation (13).

FNR solutions

A.1 *Prebuckling.*

$$N_0 = p, \qquad M_0 = 0, \qquad e_0 = -H, \qquad w_0 = H, \qquad v_0 = 0. \tag{A.1}$$

A.2 *Buckling.*

$$w_1 = A_1 \sin \mu\phi + A_2 \sin \phi + A_3 \phi \cos \phi \tag{A.2}$$

$$v_1 = -\frac{1}{\mu} A_1 \cos \mu\phi + (QA_3 - A_2) \cos \phi + A_3 \phi \sin \phi \tag{A.3}$$

where

$$\mu = \sqrt{p} \text{ and } Q = \frac{1 + H(\mu^2 - 1)}{1 - H(\mu^2 - 1)}.$$

For pinned arches the eigenvalues are determined as the roots of

$$\mu \tan \mu\alpha = \frac{(1+Q) \sin^2 \alpha}{(1 + 2Q - \mu^2 Q) \sin \alpha \cos \alpha + \alpha(1 - \mu^2)} \tag{A.4}$$

while for clamped arches

$$\mu \tan \mu\alpha = \frac{(1 + \mu^2 Q) \sin \alpha \cos \alpha - \alpha(1 - \mu^2)}{(1+Q) \cos^2 \alpha}. \tag{A.5}$$

The A_i above are determined from two of the homogeneous equations leading to the eigenvalue problem and by stipulating that the maximum buckling displacement (dimensionless) shall be equal to unity.

A.3 *Second order solution.*

$$w_2 = D_1 \cos \mu\phi + D_2 \cos \phi + D_3 \phi \sin \phi + C_0 + \sum_{i=1}^{4} C_i \cos p_i \phi \tag{A.6}$$

$$v_2 = \frac{1}{\mu} D_1 \sin \mu\phi + (D_2 + QD_3) \sin \phi - D_3 \phi \cos \phi + \sum_{i=1}^{4} C_{i+4} \sin p_i \phi \tag{A.7}$$

where $p_1 = 2$, $p_2 = 2\mu$, $p_3 = 1 + \mu$, $p_4 = 1 - \mu$. The C_k are determined for the particular solution and the D_k represent the complementary terms which are determined by satisfaction of boundary conditions.

The following order of magnitude relations are of interest, and are derivable from the details of the first and second order solutions:

$$\begin{aligned} &e_1 = 0(H), \qquad \chi_1 = 0(1), \qquad \chi_2 = 0(1) \\ &e_2 = 0(1), \qquad e_2 + \tfrac{1}{2}\chi_1^2 = 0(H). \end{aligned} \tag{A.8}$$

APPENDIX B

MCR solutions

B.1 *Prebuckling-pinned.*

$$w_0 = (1-\psi)\left[1+\frac{(\alpha\sin\alpha-2\cos\alpha)\cos\phi}{2\cos^2\alpha}-\frac{\phi\sin\phi}{2\cos\alpha}\right] \tag{B.1}$$

$$N_0 = p\psi, \qquad M_0 = -p(1-\psi)\left[1-\frac{\cos\phi}{\cos\alpha}\right] \tag{B.2}$$

$$\psi = -\frac{e_0}{H} = \left[1+H\left(\frac{2\alpha\cos^2\alpha}{\alpha(1+2\cos^2\alpha)-3\sin\alpha\cos\alpha}\right)\right]^{-1}. \tag{B.3}$$

B.2 *Prebuckling-clamped.*

$$w_0 = (1-\psi)\left[1-\frac{(\alpha\cos\alpha+\sin\alpha)\cos\phi+(\sin\alpha)\phi\sin\phi}{\alpha+\sin\alpha\cos\alpha}\right] \tag{B.4}$$

$$N_0 = p\psi, \qquad M_0 = -p(1-\psi)\left[1-\frac{2\sin\alpha}{\alpha+\sin\alpha\cos\alpha}\right] \tag{B.5}$$

$$\psi = -\frac{e_0}{H} = \left[1+H\left(\frac{\alpha^2+\alpha\sin\alpha\cos\alpha}{\alpha^2+\alpha\sin\alpha\cos\alpha-2\sin^2\alpha}\right)\right]^{-1}. \tag{B.6}$$

B.3 *Buckling.*

$$w_1 = A_1\sin\mu\phi+A_2\sin\phi+A_3\phi\cos\phi \tag{B.7}$$

$$v_1 = \frac{1-H(\mu^2-1)}{\mu}A_1\cos\mu\phi+(\bar{Q}A_3-A_2)\cos\phi+A_3\phi\sin\phi \tag{B.8}$$

where

$$\mu^2 = p_{cr}\psi(1+H) \quad\text{and}\quad \bar{Q} = \frac{1+H(\mu^2+1)}{1-H(\mu^2-1)}.$$

For pinned arches the eigenvalues are determined as the roots of

$$\mu\tan\mu\alpha = \frac{2[1-H(\mu^2-1)]\sin^2\alpha}{(2+\bar{Q}-\mu^2\bar{Q})\sin\alpha\cos\alpha+\alpha(1-\mu^2)} \tag{B.9}$$

while for clamped arches

$$\mu\tan\mu\alpha = \frac{[1+\mu^2\bar{Q}-H(\mu^2-1)]\sin\alpha\cos\alpha-\alpha(1-\mu^2)(1+H)}{(1+\bar{Q})\cos^2\alpha}. \tag{B.10}$$

The A_i above are determined from two of the homogeneous equations leading to the eigenvalue problem and by stipulating that the maximum buckling displacement (dimensionless) shall be equal to unity.

B.4 *Second order solution.*

$$w_2 = D_1 \cos \mu\phi + D_2 \cos \phi + D_3\phi \sin \phi + e_0 + \sum_{i=1}^{4} C_i \cos p_i\phi \tag{B.11}$$

$$v_2 = -\frac{1-H(\mu^2-1)}{\mu} D_1 \sin \mu\phi + (D_2 + \bar{Q}D_3) \sin \phi - D_3\phi \cos \phi + \sum_{i=1}^{4} C_{i+4} \sin p_i\phi \tag{B.12}$$

where $p_1 = 2\mu$, $p_2 = 2$, $p_3 = \mu+1$, $p_4 = \mu-1$. The C_k are determined for the particular solution and the D_k represent the complementary terms which are determined by satisfaction of boundary conditions.

The order of magnitude relations of the FNR solution are also found to hold for the MCR solution, i.e.

$$\begin{gathered} e_1 = 0(H), \qquad \chi_1 = 0(1), \qquad \chi_2 = 0(1) \\ e_2 = 0(1), \qquad e_2 + \tfrac{1}{2}\chi_1^2 = 0(H). \end{gathered} \tag{B.13}$$

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Абстракт—Исследуется линия бифуркации в процессе выпучивания и поведение после выпучивания для крутых, сжимаемых, круглых арок. Арки нагруженные равномерным постоянным направленным давлением. Они могут быть как шарнирные так и заделанные. Разработка задачи приводится на основе теории Койтера. Используются две разные теории арок, с целью исправления исследования эффекта изгиба перед началом выпучивания. Оказывается что заделанные арки всегда неустойчивы после бифуркации, по сравнению с шарнирными, которые превращаются из поведения неустойчивого к устойчивому, если арка приближается к полукруглой. Сравниваются, также, результаты с результатами, полученными из теории пологой арки. Сравнение, по рассуждению, хорошое для умеренно крутых арок. Эффект растяжимости или сжимаемости срединной поверхности и изгиба перед выпучиванием является фактически необнаруженный.